

SQUARE SUMMABILITY WITH GEOMETRIC WEIGHT FOR CLASSICAL ORTHOGONAL EXPANSIONS

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Abstract Let f_k be the k -th Fourier coefficient of a function f in terms of the orthonormal Hermite, Laguerre or Jacobi polynomials. We give necessary and sufficient conditions on f for the inequality $\sum_k |f_k|^2 \theta^k < \infty$ to hold with $\theta > 1$. As a by-product new orthogonality relations for the Hermite and Laguerre polynomials are found. The basic machinery for the proofs is provided by the theory of reproducing kernel Hilbert spaces.

Keywords: Hermite polynomials, Laguerre polynomials, Jacobi polynomials, square summability, orthogonality, reproducing kernel, Szegő space, spaces of entire functions.

1. Introduction. The goal of this paper is to find necessary and sufficient conditions to be imposed on a function f for its Fourier coefficients in terms of classical orthogonal polynomials to satisfy the inequality

$$\sum_{k=0}^{\infty} |f_k|^2 \theta^k < \infty \quad (1)$$

with $\theta > 1$. So, we have three problems corresponding to the following three definitions of f_k :

$$f_k = \int_{-\infty}^{\infty} f(x) \mathbb{H}_k(x) e^{-x^2} dx, \quad (2)$$

$$f_k = \int_0^{\infty} f(x) \mathbb{L}_k^{\nu}(x) x^{\nu} e^{-x} dx, \quad (3)$$

and

$$f_k = \int_{-1}^1 f(x) \mathbb{P}_k^{\alpha, \beta}(x) (1-x)^{\alpha} (1+x)^{\beta} dx. \quad (4)$$

Here \mathbb{H}_k , \mathbb{L}_k^{ν} and $\mathbb{P}_k^{\alpha, \beta}$ is the k -th orthonormal polynomial of Hermite, Laguerre and Jacobi, respectively [Szego91]. For the sake of convenience we use orthonormal instead of standardly normalized versions of the classical polynomials. In each case f is defined on the interval of orthogonality of the corresponding system of polynomials. We will use φ_k as a generic notation for either of the three types of polynomials.

Classes of functions with rapidly decreasing Fourier coefficients in classical orthogonal polynomials have been extensively studied. We only mention a few contributions without any attempt to make a survey. The series of papers [Hille39]-[Hille80] by E. Hille is devoted to the Hermite expansions. Among other things Hille studied expansions with f_k vanishing as quick as $\exp(-\tau(2k+1)^{1/2})$. The functions possessing such coefficients are holomorphic in the strip $|\Im z| < \tau$. Hille provided an exact description of the linear vector space with compact convergence topology formed by these functions where the set $\{\mathbb{H}_k : k \in \mathbb{N} \cup 0\}$ is a basis. Its members are characterized by a suitable growth condition. In addition, he studied the convergence on and analytic continuation through the boundary of the strip.

The convergence domain of the Laguerre series with $f_k \sim \exp(-\tau k^{1/2})$ is the interior of the parabola $\Re(-z)^{1/2} = \tau/2$. Rusev in [Rusev84] described the linear vector space with compact convergence topology formed by functions holomorphic in $\{z : \Re(-z)^{1/2} < \tau/2\}$ where the set $\{\mathbb{L}_k^\nu : k \in \mathbb{N} \cup 0\}$ is a basis, and Boyadjiev in [Boyad92] studied the behavior of the Laguerre series on the boundary of the convergence domain.

If coefficients (2) or (3) decrease as fast as $\exp(-\tau k^\eta)$ with $\eta > 1/2$ the function f is entire. The spaces comprising such functions for Hermite expansions have been characterized by Janssen and van Eijndhoven in [JanEijnd90]. If the Fourier-Hermite coefficients f_k decline quicker than any geometric progression, the suitable characterization is provided by Berezanskij and Kondratiev in [BerKondr95] (see Corollary 1.1 below). Byun was the first to study the Hermite expansions with condition (1) - see remark after Theorem 1. For the Laguerre expansions with $\limsup_k |f_k|^{1/k} < 1$ Zayed related the singularities of the Borel transform of f with those of $F(z) = \sum_k f_k z^k$ in [Zayed81].

For the Legendre expansions with $\limsup_k |f_k|^{1/k} = \zeta < 1$ Nehari relates the singularities of f on the boundary of convergence domain $\{z : |z+1| + |z-1| < \zeta + \zeta^{-1}\}$ to those of $F(z) = \sum_k f_k z^k$ in [Nehari56]. Gilbert in [Gilbert64] and Gilbert and Howard in [GilbHoward66] generalized the results of Nehari to the Gegenbauer expansions and to expansions in eigenfunctions of a Sturm-Liouville operator.

Functions satisfying (1) apparently form a proper subclass of functions with $\limsup_k |f_k|^{1/k} \leq \theta^{-1/2}$. On the other hand the condition (1) itself cannot be expressed in terms of asymptotics of f_k . Consequently, our criteria for the validity of (1) are of different character from those contained in the above references. Although they also describe the growth of f for the Hermite and Laguerre expansions and its boundary behaviour for the Jacobi expansions, our growth conditions are given in terms of existence of certain weighted area integrals of f and cannot be expressed by an estimate of the modulus, while the restriction on the boundary behaviour is given on the whole boundary and not in terms of analysis of individual singularities.

2. Preliminaries. Throughout the paper the following standard notation will be used: \mathbb{N} , \mathbf{R} , \mathbf{R}^+ and \mathbb{C} will denote positive integers, real numbers, positive real numbers and the finite complex plane, respectively. Since coefficients (2)-(4) do not change if we modify f on a set of Lebesgue measure zero, all our statements about the properties of f should be understood to hold almost everywhere. If, for instance, we say that f is the restriction of a holomorphic function to (some part of) the real axis, it means that f is allowed to differ from such restriction on a set of zero measure.

All proofs in the paper hinge on the theory of reproducing kernel Hilbert spaces (RKHS), so for convenience we briefly outline the basic facts of the theory we will make use of.

For a Hilbert space H comprising complex-valued functions on a set E , the reproducing kernel $K(p, q) : E \times E \rightarrow \mathbb{C}$ is a function that belongs to H as a function of p for every fixed $q \in E$ and possesses the reproducing property

$$f(q) = (f, K(\cdot, q))_H$$

for every $f \in H$ and for any $q \in E$. If a Hilbert space admits the reproducing kernel then this kernel is unique and positive definite on $E \times E$:

$$\sum_{i,j=1}^n K(p_i, p_j) c_i \overline{c_j} \geq 0 \quad (5)$$

for an arbitrary finite complex sequence $\{c_i\}$ and any points $p_i \in E$. The theorem of Moore and Aronsjain [Aron50] states that the converse is also true: every positive definite kernel K on $E \times E$ uniquely determines a Hilbert space H admitting K as its reproducing kernel. This fact justifies the notation H_K for the Hilbert space H induced by the kernel K . The following propositions can be found in [Aron50, Saitoh97].

Proposition 1 *If H_K is a Hilbert space of functions $E \rightarrow \mathbb{C}$ and s is an arbitrary non-vanishing function on E , then*

$$K_s(p, q) = s(p)\overline{s(q)}K(p, q) \quad (6)$$

is the reproducing kernel of the Hilbert space H_{K_s} comprising all functions on E expressible in the form $f_s(p) = s(p)f(p)$ with $f \in H_K$ and equipped with inner product

$$(f_s, g_s)_{H_{K_s}} = \left(\frac{f_s}{s}, \frac{g_s}{s} \right)_{H_K}. \quad (7)$$

Proposition 2 *Let $E_1 \subset E$ and K_1 be the restriction of a positive definite kernel K to $E_1 \times E_1$. Then the RKHS H_{K_1} comprises all restrictions to E_1 of functions from H_K and has the norm given by*

$$\|f_1\|_{H_{K_1}} = \min \{ \|f\|_{H_K}; f|_{E_1} = f_1, f \in H_K \}. \quad (8)$$

Proposition 3 *If RKHS H_K is separable and $\{\psi_k : k \in \mathbb{N}\}$ is a complete orthonormal system in H_K , then its reproducing kernel is expressed by*

$$K(p, q) = \sum_k \psi_k(p)\overline{\psi_k(q)}, \quad (9)$$

where the series (9) converges absolutely for all $p, q \in E$ and uniformly on every subset of E , where $K(q, q)$ is bounded.

The relation $K_1 \ll K_2$ will mean that $K_2 - K_1$ is positive definite. This relation introduces partial ordering into the set of positive definite kernels on $E \times E$. Inclusion $H_{K_1} \subset H_{K_2}$ and equality $H_{K_1} = H_{K_2}$ will be understood in the set-theoretic sense, which implies, however, that the same relations hold in the topological sense as stated in the following two propositions.

Proposition 4 *Inclusion $H_{K_1} \subset H_{K_2}$ takes place iff $K_1 \ll MK_2$ for a constant $M > 0$. In this case $M^{1/2}\|f\|_1 \geq \|f\|_2$ for all $f \in H_{K_1}$. (Here $\|f\|_1$ and $\|f\|_2$ are the norms in H_{K_1} and H_{K_2} , respectively).*

Proposition 5 *Equality $H_{K_1} = H_{K_2}$ takes place iff $mK_2 \ll K_1 \ll MK_2$ for some positive constants m, M . In this case $m^{1/2}\|f\|_1 \leq \|f\|_2 \leq M^{1/2}\|f\|_1$.*

The kernels satisfying Proposition 5 are said to be equivalent which is denoted by $K_1 \approx K_2$.

It is shown in [Aron50] that the RKHS H_K induced by the kernel $K(p, q) = K_1(p, q)K_2(p, q)$ consists of all restrictions to the diagonal of $E' = E \times E$ (i.e. the set of points of the form $\{p, p\}$) of the elements of the tensor product $H' = H_{K_1} \otimes H_{K_2}$. The space H_K is characterized by

Proposition 6 *Let $K(p, q) = K_1(p, q)K_2(p, q)$ and let $\{\psi_k : k \in \mathbb{N}\}$ be a complete orthonormal set in H_{K_2} . Then the RKHS H_K comprises the functions of the form*

$$f(p) = \sum_{k=1}^{\infty} f_k^1(p)\psi_k(p), \quad f_k^1 \in H_{K_1}, \quad \sum_{k=1}^{\infty} \|f_k^1\|_1^2 < \infty. \quad (10)$$

The norm in H_K is given by

$$\|f\|_{H_K}^2 = \min \left\{ \sum_{k=1}^{\infty} \|f_k^1\|_1^2 \right\},$$

where the minimum is taken over all representations of f in the form (10) and is attained on one such representation.

3. Results for the Hermite and Laguerre expansions. Functions satisfying (1) form a Hilbert space with inner product

$$(f, g) = \sum_{k=0}^{\infty} f_k \overline{g_k} \theta^k. \quad (11)$$

This space will be denoted by \mathcal{H}_θ for the Hermite expansions and by \mathcal{L}_θ^ν for the Laguerre expansions. The sets $\{\theta^{-k/2} \mathbb{H}_k\}_{k \in \mathbb{N} \cup 0}$ and $\{\theta^{-k/2} \mathbb{L}_k^\nu\}_{k \in \mathbb{N} \cup 0}$ constitute orthonormal bases of the spaces \mathcal{H}_θ and \mathcal{L}_θ^ν , respectively. For each space we can form the reproducing kernel according to (9):

$$K(z, \bar{u}) = \sum_k \varphi_k(z) \overline{\varphi_k(u)} \theta^{-k}. \quad (12)$$

The explicit formulae for these kernels are known to be [Bateman53]:

$$HK_\theta(z, \bar{u}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathbb{H}_k(z) \overline{\mathbb{H}_k(u)} \theta^{-k} = \frac{\theta}{\sqrt{\pi(\theta^2 - 1)}} \exp\left(\frac{2z\bar{u}\theta - z^2 - \bar{u}^2}{\theta^2 - 1}\right) \quad (13)$$

(Mehler's formula) and

$$LK_\theta^\nu(z, \bar{u}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathbb{L}_k^\nu(z) \overline{\mathbb{L}_k^\nu(u)} \theta^{-k} = \frac{\theta^{\nu/2+1}}{\theta - 1} \exp\left(-\frac{z + \bar{u}}{\theta - 1}\right) (z\bar{u})^{-\frac{\nu}{2}} I_\nu\left(2\frac{\sqrt{\theta z\bar{u}}}{\theta - 1}\right) \quad (14)$$

(Hardy-Hille's formula). Here I_ν is the modified Bessel function. The kernels (13) and (14) are entire functions of both z and \bar{u} . The space \mathcal{H}_θ comprises functions on \mathbf{R} , while the space \mathcal{L}_θ^ν comprises functions on \mathbf{R}^+ . Hence, we consider the restrictions of the kernels (13) and (14) to \mathbf{R} and \mathbf{R}^+ , respectively. Applying Proposition 2 with $E = \mathbb{C}$ and $E' = \mathbf{R}$ or $E' = \mathbf{R}^+$ we conclude that the spaces \mathcal{H}_θ and \mathcal{L}_θ^ν are formed by all restrictions to \mathbf{R} and \mathbf{R}^+ , respectively, of entire functions from the spaces generated by the kernels (13) and (14). We can drop minimum in (8) due to uniqueness of analytic extension and, consequently, the norm induced by inner product (11) equals the norm in H_{HK_θ} or in $H_{LK_\theta^\nu}$.

Next, we observe that both kernels (13) and (14) are of the form $s(z)\overline{s(u)}K(z\bar{u})$ with non-vanishing functions $s_H(z) = e^{-z^2/(\theta^2-1)}$ and $s_L(z) = e^{-z/(\theta-1)}$. Hence we are in the position to apply Proposition 1. In compliance with (7) the norms in H_{HK_θ} and $H_{LK_\theta^\nu}$ are known once we have found the norms in the spaces induced by the kernels

$$\widetilde{HK}_\theta(z\bar{u}) = \frac{\theta}{\sqrt{\pi(\theta^2 - 1)}} \exp\left(\frac{2z\bar{u}\theta}{\theta^2 - 1}\right) \quad (15)$$

and

$$\widetilde{LK}_\theta^\nu(z\bar{u}) = \frac{\theta^{\nu/2+1}}{\theta - 1} (z\bar{u})^{-\frac{\nu}{2}} I_\nu\left(2\frac{\sqrt{\theta z\bar{u}}}{\theta - 1}\right). \quad (16)$$

Both of them depend on the product $z\bar{u}$ and are rotation invariant thus. Rotation invariance of the kernel implies radial symmetry of the measure with respect to which the integral representing the norm is taken. For the kernel (15) the measure and the space are well-known. It is the Fischer-Fock (or the Bargmann-Fock) space \mathcal{F}_θ of entire functions with finite norms

$$\|f\|_{\mathcal{F}_\theta}^2 = \frac{2}{\sqrt{\pi(\theta^2 - 1)}} \int_{\mathbb{C}} |f(z)|^2 \exp\left(-\frac{2\theta|z|^2}{\theta^2 - 1}\right) d\sigma, \quad (17)$$

where the integration is with respect to Lebesgue's area measure. By Proposition 1 the final result for the Hermite expansions now becomes straightforward.

Theorem 1 *Inequality (1) with f_k defined by (2) holds true for all restrictions to \mathbf{R} of the entire functions with*

$$\int_{\mathbb{C}} |f(z)|^2 \exp\left[-2\left(\frac{(\Re z)^2}{\theta + 1} + \frac{(\Im z)^2}{\theta - 1}\right)\right] d\sigma < \infty \quad (18)$$

and only for them.

For inner product (11) this leads to the expression

$$(f, g)_{\mathcal{H}_\theta} = (f, g)_{H_{HK_\theta}} = \frac{2}{\sqrt{\pi(\theta^2 - 1)}} \int_{\mathbb{C}} f(z) \overline{g(z)} \exp \left[-2 \left(\frac{(\Re z)^2}{\theta + 1} + \frac{(\Im z)^2}{\theta - 1} \right) \right] d\sigma. \quad (19)$$

Since polynomials \mathbb{H}_k are orthogonal with respect to this inner product, we obtain the following orthogonality relation for the standardly normalized Hermite polynomials \mathbb{H}_k :

$$\frac{2}{\sqrt{(\theta^2 - 1)}} \int_{\mathbb{C}} \mathbb{H}_k(z) \overline{\mathbb{H}_m(z)} \exp \left[-\frac{2(\Re z)^2}{\theta + 1} - \frac{2(\Im z)^2}{\theta - 1} \right] d\sigma = \delta_{k,m} \pi (2\theta)^k k!. \quad (20)$$

These results for the Hermite expansions have been essentially proved by Du-Wong Byun in [Byun93], although the emphasis in his work is different and it seems that the orthogonality relation (20) has not been noticed.

The following corollary is an immediate consequence of (18).

Corollary 1.1 *Inequality (1) with f_k defined by (2) holds true for **all** $\theta > 1$ iff f is the restriction to \mathbf{R} of an entire function F satisfying $|F(z)| \leq C e^{\varepsilon|z|^2}$ for all $\varepsilon > 0$ and a constant $C = C(\varepsilon)$ independent of z .*

A direct proof is given in [BerKondr95].

For the Laguerre expansions the situation is a bit more complicated. The kernel (16) is a particular case of a much more general hypergeometric kernel. The spaces generated by hypergeometric kernels are studied in depth in [Karp03]. For the kernel (16) we get the space of entire functions with finite norms

$$\|f\|_{\widetilde{LK}_\theta^\nu}^2 = \frac{2\theta^{-\nu/2}}{\pi(\theta - 1)} \int_{\mathbb{C}} |f(z)|^2 |z|^\nu K_\nu \left(\frac{2\sqrt{\theta}|z|}{\theta - 1} \right) d\sigma, \quad (21)$$

where K_ν is the modified Bessel function of the second kind (or the McDonald function). Application of Proposition 1 brings us to our final result for the Laguerre expansions.

Theorem 2 *Inequality (1) with f_k defined by (3) holds true for all restrictions to \mathbf{R}^+ of the entire functions with*

$$\int_{\mathbb{C}} |f(z)|^2 \exp \left(\frac{2\Re z}{\theta - 1} \right) |z|^\nu K_\nu \left(\frac{2\sqrt{\theta}|z|}{\theta - 1} \right) d\sigma < \infty \quad (22)$$

and only for them.

The orthogonality relation for the standardly normalized Laguerre polynomials L_k^ν that follows from this result is given by

$$\frac{2\theta^{-\nu/2}}{\pi(\theta - 1)} \int_{\mathbb{C}} L_k^\nu(z) \overline{L_m^\nu(z)} \exp \left(\frac{2\Re z}{\theta - 1} \right) |z|^\nu K_\nu \left(\frac{2\sqrt{\theta}|z|}{\theta - 1} \right) d\sigma = \delta_{k,m} \frac{\Gamma(k + \nu + 1) \theta^k}{k!}. \quad (23)$$

Corollary 2.1 *Inequality (1) with f_k defined by (3) holds true for **all** $\theta > 1$ iff f is the restriction to \mathbf{R}^+ of an entire function F satisfying $|F(z)| \leq C e^{\varepsilon|z|}$ for all $\varepsilon > 0$ and a constant $C = C(\varepsilon)$ independent of z .*

This corollary can be easily derived from (22) with the help of asymptotic relation [Bateman53]

$$K_\nu(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \rightarrow \infty.$$

4. Results for the Jacobi expansions. The space of complex-valued functions on $(-1, 1)$ whose Fourier-Jacobi coefficients (4) satisfy (1) will be denoted by $\mathcal{J}_\theta^{\alpha, \beta}$. Pursuing the same line of argument as in the previous section, we form the reproducing kernel of this space found by Baily's formula [Bateman53]

$$JK_{\alpha, \beta}^\theta(z, \bar{u}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathbb{P}_k^{\alpha, \beta}(z) \overline{\mathbb{P}_k^{\alpha, \beta}(u)} \frac{1}{\theta^k} = \frac{\theta^{\alpha+\beta+1}(\theta-1)}{\tau(\alpha, \beta)(\theta+1)^{\alpha+\beta+2}} \times \\ \times F_4\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1, \frac{\alpha}{2} + \frac{\beta}{2} + \frac{3}{2}; \alpha+1, \beta+1; \frac{\theta(1-z)(1-\bar{u})}{(\theta+1)^2}, \frac{\theta(1+z)(1+\bar{u})}{(\theta+1)^2}\right), \quad (24)$$

where

$$F_4(a, b; c, c'; t, s) = \sum_{m, n=0}^{\infty} \frac{(a)_{n+m} (b)_{n+m}}{(c)_n (c')_m m! n!} t^m s^n \quad (25)$$

is Appel's hypergeometric function and $\tau(\alpha, \beta) = 2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) / \Gamma(\alpha+\beta+2)$. Define the ellipse E_θ by

$$E_\theta = \{z : |z-1| + |z+1| < \theta^{1/2} + \theta^{-1/2}\}. \quad (26)$$

Our first observation here is that both the series on the right hand side and on the left hand side of (24) converge absolutely and uniformly on compact subsets of $E_\theta \times E_\theta$ (see [Karp00]). The implication of the uniform convergence is holomorphy of the kernel (24) in $E_\theta \times E_\theta$ with respect to both variables. Application of Proposition 2 with $E = E_\theta$, $E' = (-1, 1)$ leads to the assertion that the space $\mathcal{J}_\theta^{\alpha, \beta}$ is formed by restrictions to the interval $(-1, 1)$ of functions holomorphic in E_θ thereby the norm in $\mathcal{J}_\theta^{\alpha, \beta}$ equals the norm in $H_{JK_{\alpha, \beta}^\theta}$ due to uniqueness of analytic continuation.

Our main result for the Jacobi expansions will be derived from its particular case $\alpha = \beta = \lambda - 1/2$ thereby the orthonormal Jacobi polynomials reduce to the orthonormal Gegenbauer polynomials \mathbb{C}_k^λ . The reproducing kernel (24) reduces in this case to

$$GK_\lambda^\theta(z, \bar{u}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathbb{C}_k^\lambda(z) \overline{\mathbb{C}_k^\lambda(u)} \frac{1}{\theta^k} = \\ = \frac{\theta^{2\lambda}(\theta^2 - 1)}{\tau(\lambda)(\theta^2 - 2\theta z \bar{u} + 1)^{\lambda+1}} {}_2F_1\left(\frac{\lambda+1}{2}, \frac{\lambda+2}{2}; \lambda + \frac{1}{2}; \frac{4\theta^2(1-z^2)(1-\bar{u}^2)}{(\theta^2 - 2\theta z \bar{u} + 1)^2}\right), \quad (27)$$

where ${}_2F_1$ is the Gauss hypergeometric function and $\tau(\lambda) = \sqrt{\pi} \Gamma(\lambda+1/2) / \Gamma(\lambda+1)$. The series on both sides of (27) again converge absolutely and uniformly on compact subsets of $E_\theta \times E_\theta$. Formula (27) can be obtained from (24) by using a reduction formula for F_4 and applying a quadratic transformation to the resulting hypergeometric function. Details are in [Karp00].

Let ∂E_θ denote the boundary of the ellipse E_θ . We introduce the weighted Szegő space $AL_2(\partial E_\theta; \rho)$ with continuous positive weight $\rho(z)$ defined on ∂E_θ as the set of functions holomorphic in E_θ , possessing non-tangential boundary values almost everywhere on ∂E_θ and having finite norms

$$\|f\|_{AL_2(\partial E_\theta; \rho)}^2 = \int_{\partial E_\theta} |f(z)|^2 \rho(z) |dz| < \infty. \quad (28)$$

We will write $AL_2(\partial E_\theta)$ for $AL_2(\partial E_\theta; 1)$.

For $\lambda = 0$, the orthonormal Gegenbauer polynomials reduce to the orthonormal Chebyshev polynomials of the first kind \mathbb{T}_k :

$$\mathbb{C}_k^0(z) = \mathbb{T}_k(z) = (2/\pi)^{\frac{1}{2}} \mathbb{T}_k(z) = (2/\pi)^{\frac{1}{2}} \cos(k \arccos z), \quad k \in \mathbb{N}, \\ \mathbb{C}_0^0(z) = \mathbb{T}_0(z) = (1/\pi)^{\frac{1}{2}} \mathbb{T}_0(z) = (1/\pi)^{\frac{1}{2}}, \quad (29)$$

where \mathbb{T}_k is the k -th Chebyshev polynomial of the first kind in standard normalization.

Lemma 1 *Polynomials $(\theta^k + \theta^{-k})^{-1/2} \mathbb{T}_k/2$ form orthonormal basis of the space $AL_2(\partial E_\theta; |z^2 - 1|^{-\frac{1}{2}})$.*

Proof. The boundary ∂E_θ is an analytic arc which implies the completeness of the set of all polynomials in $AL_2(\partial E_\theta; |z^2 - 1|^{-\frac{1}{2}})$ (see for instance [Gaier80]). To prove orthogonality we will need some properties of the Zhukowskii function $z = (w + w^{-1})/2$. This function maps the annulus $1 < |w| < \sqrt{\theta}$ one-to-one and conformally onto E_θ cut along the interval $(-1, 1)$ thereby the circle $|w| = \sqrt{\theta}$ corresponds to ∂E_θ . The inverse function is given by $w = z + \sqrt{z^2 - 1}$, where the principal value of the square root is to be chosen. We see by differentiation that the infinitesimal arc lengths are connected by the relation $|dz| = |w^2 - 1| |dw| / (2|w|^2)$. Note also that $z^2 - 1 = (w^2 - 1)^2 / (4w^2)$. Applying the identity $\mathbb{T}_k(z(w)) = w^k + w^{-k}$ we get by the substitution $z = (w + w^{-1})/2$:

$$\begin{aligned} & \frac{1}{2\pi} \int_{\partial E_\theta} \mathbb{T}_k(z) \overline{\mathbb{T}_m(z)} |z^2 - 1|^{-\frac{1}{2}} |dz| = \frac{1}{2\pi} \int_{|w|=\sqrt{\theta}} \left(w^k + \frac{1}{w^k} \right) \left(\overline{w^m} + \frac{1}{\overline{w^m}} \right) \frac{|dw|}{|w|} = \\ & = \frac{\theta^{(k+m)/2}}{2\pi} \int_0^{2\pi} e^{i\varphi(k-m)} d\varphi + \frac{\theta^{-(k+m)/2}}{2\pi} \int_0^{2\pi} e^{i\varphi(m-k)} d\varphi + \frac{\theta^{(k-m)/2}}{2\pi} \int_0^{2\pi} e^{i\varphi(k+m)} d\varphi + \\ & \quad + \frac{\theta^{(m-k)/2}}{2\pi} \int_0^{2\pi} e^{-i\varphi(k+m)} d\varphi = \begin{cases} 0, & k \neq m, \\ \theta^k + \theta^{-k}, & k = m \neq 0, \\ 4, & k = m = 0. \end{cases} \end{aligned}$$

Combined with (29) this proves the lemma. \square

Denote $\mathcal{G}_\theta^\lambda = \mathcal{J}_\theta^{\lambda-1/2, \lambda-1/2}$. We are ready to formulate our main result for the Gegenbauer expansions.

Theorem 3 *Let $\lambda \geq 0$. The space $\mathcal{G}_\theta^\lambda$ is formed by all restrictions of the elements of $AL_2(\partial E_\theta)$ to the interval $(-1, 1)$. The norms in $\mathcal{G}_\theta^\lambda$ and $AL_2(\partial E_\theta)$ are equivalent.*

Proof. The proof will be divided in three steps.

Step 1. For the space $H_{GK_0^\theta}$ induced by the kernel (27) with $\lambda = 0$ we want to prove that

$$H_{GK_0^\theta} = AL_2(\partial E_\theta). \quad (30)$$

The weight $|z^2 - 1|^{-1/2}$ is positive and continuous on ∂E_θ so the norms in $AL_2(\partial E_\theta; |z^2 - 1|^{-\frac{1}{2}})$ and $AL_2(\partial E_\theta)$ are equivalent and these spaces coincide elementwise. According to Lemma 1 and formula (9) the space $AL_2(\partial E_\theta; |z^2 - 1|^{-\frac{1}{2}})$ admits the reproducing kernel given by

$$R_\theta(z, \bar{u}) = \frac{1}{4} \sum_{k=0}^{\infty} \frac{\mathbb{T}_k(z) \overline{\mathbb{T}_k(u)}}{\theta^k + \theta^{-k}}.$$

This kernel is equivalent to the kernel GK_0^θ due to (29) and inequalities

$$\frac{1}{2} \sum_{i,j=0}^n GK_0^\theta(z_i, \bar{z}_j) c_i \bar{c}_j \leq \sum_{i,j=0}^n \sum_{k=0}^{\infty} \frac{\mathbb{T}_k(z_i) \overline{\mathbb{T}_k(z_j)}}{\theta^k + \theta^{-k}} c_i \bar{c}_j \leq \sum_{i,j=0}^n GK_0^\theta(z_i, \bar{z}_j) c_i \bar{c}_j$$

satisfied for any choice of $n \in \mathbb{N}$, $c_i \in \mathbb{C}$ and $z_i \in E_\theta$. Hence by Proposition 5 our claim is proved.

Step 2. Consider the following auxiliary kernel:

$$\hat{K}_\lambda^\theta(z, \bar{u}) = \frac{\pi^{-1}(\theta^2 - 1)}{\theta^2 - 2\theta z \bar{u} + 1} {}_2F_1 \left(\frac{\lambda + 1}{2}, \frac{\lambda + 2}{2}; \lambda + \frac{1}{2}; \frac{4\theta^2(1 - z^2)(1 - \bar{u}^2)}{(\theta^2 - 2\theta z \bar{u} + 1)^2} \right). \quad (31)$$

It is positive definite as will be shown below. Substitution $\lambda = 0$ yields the identity

$$\hat{K}_0^\theta(z, \bar{u}) = GK_0^\theta(z, \bar{u}). \quad (32)$$

We want to prove that for all $\lambda, \mu > -\frac{1}{2}$

$$H_{\hat{K}_\lambda^\theta} = H_{\hat{K}_\mu^\theta}. \quad (33)$$

According to Proposition 5 we need to show that $\hat{K}_\lambda^\theta \approx \hat{K}_\mu^\theta$. Following the definition of the positive definite kernel (5), choose $n \in \mathbb{N}$, a finite complex sequence c_i and points $z_i \in E_\theta$, $i = \overline{1, n}$. Positive definiteness of the kernel $[4\theta^2(1-z^2)(1-\bar{u}^2)]^k/(\theta^2-2\theta z\bar{u}+1)^{2k+1}$ due to its reproducing property in the Hilbert space of functions representable in the form $f(z) = (1-z^2)^k g(z)$, where g belongs to the Bergman-Selberg space generated by the kernel (36) with $\lambda = 2k+1$, and interchange of the order of summations justified by absolute convergence, lead to the estimates

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n \hat{K}_\mu^\theta(z_i, \bar{z}_j) c_i \bar{c}_j = \frac{\theta^2-1}{\pi} \sum_{k=0}^{\infty} a_k^\mu \sum_{i,j=1}^n \frac{[4\theta^2(1-z_i^2)(1-\bar{z}_j^2)]^k}{(\theta^2-2\theta z_i \bar{z}_j + 1)^{2k+1}} c_i \bar{c}_j \\ &\leq \frac{\theta^2-1}{\pi} \sup_{k \in \mathbb{N}_0} \left\{ \frac{a_k^\mu}{a_k^\lambda} \right\} \sum_{k=0}^{\infty} a_k^\lambda \sum_{i,j=1}^n \frac{[4\theta^2(1-z_i^2)(1-\bar{z}_j^2)]^k}{(\theta^2-2\theta z_i \bar{z}_j + 1)^{2k+1}} c_i \bar{c}_j = \sup_{k \in \mathbb{N}_0} \left\{ \frac{a_k^\mu}{a_k^\lambda} \right\} \sum_{i,j=1}^n \hat{K}_\lambda^\theta(z_i, \bar{z}_j) c_i \bar{c}_j, \end{aligned}$$

where

$$a_k^\lambda = \frac{([\lambda+1]/2)_k ([\lambda+2]/2)_k}{(\lambda+1/2)_k k!} = \frac{\Gamma(\lambda+1/2) \Gamma((\lambda+1)/2+k) \Gamma((\lambda+2)/2+k)}{\Gamma((\lambda+1)/2) \Gamma((\lambda+2)/2) \Gamma(\lambda+1/2+k) k!} > 0.$$

This shows the positive definiteness of the kernel \hat{K}_μ^θ . Using the asymptotic relation [Bateman53]

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} = z^{a-b} (1 + O(z^{-1})), \quad |z| \rightarrow \infty, \quad |\arg z| < \pi, \quad (34)$$

we obtain

$$\lim_{k \rightarrow \infty} \frac{a_k^\mu}{a_k^\lambda} = 1 \Rightarrow 0 < \sup_{k \in \mathbb{N}_0} \left\{ \frac{a_k^\mu}{a_k^\lambda} \right\} < \infty.$$

The estimate from below is obtained in the same fashion with $\sup_{k \in \mathbb{N}_0} \{a_k^\mu/a_k^\lambda\}$ substituted by $\inf_{k \in \mathbb{N}_0} \{a_k^\mu/a_k^\lambda\}$.

This proves equality (33). Combined with (32) and (30) this gives:

$$H_{\hat{K}_\lambda^\theta} = AL_2(\partial E_\theta) \quad (35)$$

for all $\lambda > -1/2$.

Step 3. According to (27) and (31), the kernel GK_λ^θ is related to the kernel \hat{K}_λ^θ by

$$GK_\lambda^\theta(z, \bar{u}) = B_\lambda^\theta(z, \bar{u}) \hat{K}_\lambda^\theta(z, \bar{u}),$$

where

$$B_\lambda^\theta(z, \bar{u}) = \frac{\pi \theta^{2\lambda}}{\tau(\lambda)(\theta^2 - 2\theta z \bar{u} + 1)^\lambda} = \frac{h_\lambda^\theta}{(1 - \frac{2\theta}{\theta^2+1} z \bar{u})^\lambda}, \quad h_\lambda^\theta = \frac{\pi \theta^{2\lambda}}{\tau(\lambda)(\theta^2 + 1)^\lambda}. \quad (36)$$

For $\lambda > 0$ the function $B_\lambda^\theta(z, \bar{u})$ is the reproducing kernel of the Bergman-Selberg space $H_{B_\lambda^\theta}$ [Saitoh97]. This space comprises functions holomorphic in the disk $|z| < [(\theta^2+1)/2\theta]^{1/2}$ and having finite norms

$$\|f\|_{H_{B_\lambda^\theta}} = \left[\frac{1}{h_\lambda^\theta} \sum_{k=0}^{\infty} |f_k|^2 \frac{k!}{(\lambda)_k} \left[\frac{\theta^2+1}{2\theta} \right]^k \right]^{\frac{1}{2}},$$

where f_k is the k -th Taylor coefficient of f . The functions

$$\gamma_k(z) \stackrel{\text{def}}{=} [h_\lambda^\theta]^{\frac{1}{2}} \left[\frac{(\lambda)_k}{k!} \right]^{\frac{1}{2}} \left[\frac{2\theta}{\theta^2+1} \right]^{\frac{k}{2}} z^k$$

constitute a complete orthonormal system in $H_{B_\lambda^\theta}$. Note further that the closed ellipse $\overline{E_\theta}$ is contained in the disk $|z| < [(\theta^2+1)/2\theta]^{1/2}$ due to inequality $\sqrt{(\theta^2+1)/(2\theta)} > (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})/2$, the right hand

side of which equals the big semi-axis of the ellipse E_θ . As stated in Proposition 6 the space $H_{GK_\lambda^\theta}$ is obtained by restricting the elements of the tensor product $H_{B_\lambda^\theta} \otimes H_{\hat{K}_\lambda^\theta}$ to the diagonal of $E_\theta \times E_\theta$ and comprises the functions of the form

$$f(z) = \sum_{k=1}^{\infty} g_k(z) \gamma_k(z), \quad g_k \in H_{\hat{K}_\lambda^\theta}, \quad \sum_{k=1}^{\infty} \|g_k\|_{H_{\hat{K}_\lambda^\theta}}^2 < \infty.$$

By (35) we can put $AL_2(\partial E_\theta)$ instead of $H_{\hat{K}_\lambda^\theta}$ here. For any $g \in AL_2(\partial E_\theta)$ consider the estimate

$$\|g\gamma_k\|_{AL_2(\partial E_\theta)}^2 = \int_{\partial E_\theta} |g(z)\gamma_k(z)|^2 |dz| \leq \max_{z \in \partial E_\theta} |\gamma_k(z)|^2 \|g\|_{AL_2(\partial E_\theta)}^2, \quad (37)$$

which shows that every product $g\gamma_k$ belongs to $AL_2(\partial E_\theta)$ and hence so does a finite sum of such products. Denote

$$\begin{aligned} \alpha_\lambda^\theta(k) &\stackrel{\text{def}}{=} \max_{z \in \partial E_\theta} |\gamma_k(z)| = [h_\lambda^\theta]^{\frac{1}{2}} \left[\frac{(\lambda)_k}{k!} \right]^{\frac{1}{2}} \left[\frac{2\theta}{\theta^2 + 1} \right]^{\frac{k}{2}} \left[\frac{\theta + 1}{2\sqrt{\theta}} \right]^k = \\ &= [h_\lambda^\theta]^{\frac{1}{2}} \left[\frac{(\lambda)_k}{k!} \right]^{\frac{1}{2}} \left[\frac{\theta + 1}{\sqrt{2(\theta^2 + 1)}} \right]^k, \quad \frac{\theta + 1}{\sqrt{2(\theta^2 + 1)}} < 1. \end{aligned}$$

The sequence

$$S_n(z) = \sum_{k=1}^n g_k(z) \gamma_k(z), \quad g_k \in AL_2(\partial E_\theta), \quad \sum_{k=1}^{\infty} \|g_k\|_{AL_2(\partial E_\theta)}^2 < \infty,$$

is a Cauchy sequence in $AL_2(\partial E_\theta)$. Indeed, using (37) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \|S_M - S_N\|_{AL_2(\partial E_\theta)}^2 &= \left\| \sum_{k=N}^M g_k \gamma_k \right\|_{AL_2(\partial E_\theta)}^2 = \sum_{k=N}^M \sum_{l=N}^M (g_k \gamma_k, g_l \gamma_l)_{AL_2(\partial E_\theta)} \leq \\ &\leq \sum_{k=N}^M \sum_{l=N}^M \|g_k \gamma_k\|_{AL_2(\partial E_\theta)} \|g_l \gamma_l\|_{AL_2(\partial E_\theta)} \leq \sum_{k=N}^M \sum_{l=N}^M \|g_k\|_{AL_2(\partial E_\theta)} \alpha_\lambda^\theta(k) \|g_l\|_{AL_2(\partial E_\theta)} \alpha_\lambda^\theta(l) = \\ &= \left[\sum_{k=N}^M \|g_k\|_{AL_2(\partial E_\theta)} \alpha_\lambda^\theta(k) \right]^2 \leq \sum_{k=N}^M \|g_k\|_{AL_2(\partial E_\theta)}^2 \sum_{k=N}^M [\alpha_\lambda^\theta(k)]^2. \end{aligned}$$

Since both $\sum_k \|g_k\|_{AL_2(\partial E_\theta)}^2$ and $\sum_k [\alpha_\lambda^\theta(k)]^2$ converge, the above estimates prove that the sequence S_n is Cauchy. It follows that $H_{GK_\lambda^\theta} \subset AL_2(\partial E_\theta)$. Inverse inclusion $AL_2(\partial E_\theta) \subset H_{GK_\lambda^\theta}$ is obvious, since $I(z) \equiv 1$ belongs to $H_{B_\lambda^\theta}$ and so for any $g \in AL_2(\partial E_\theta)$, the product $Ig = g \in H_{GK_\lambda^\theta}$. \square

Now it is not difficult to establish our main result for Jacobi expansions.

Theorem 4 *Let $\alpha, \beta \geq -\frac{1}{2}$. Inequality (1) with f_k defined by (4) holds true for all restrictions to the interval $(-1, 1)$ of the elements of $AL_2(\partial E_\theta)$ and only for them.*

Proof. Choose $\gamma > \max\{\alpha, \beta\}$, then

$$JK_{\alpha, \beta}^\theta(z, \bar{u}) \ll M JK_{\gamma, \gamma}^\theta(z, \bar{u})$$

for some constant $M > 0$. Indeed, for an arbitrary $n \in \mathbb{N}$, complex numbers c_i and points $z_i \in E_\theta$, $i = \overline{1, n}$, estimate using (24):

$$\sum_{i, j=1}^n JK_{\alpha, \beta}^\theta(z_i, \bar{z}_j) c_i \bar{c}_j = \sum_{k, l=0}^{\infty} a_{k, l}^{\alpha, \beta} \sum_{i, j=1}^n \frac{\theta^{k+l} (1 - z_i)^k (1 - \bar{z}_j)^k (1 + z_i)^l (1 + \bar{z}_j)^l}{(\theta + 1)^{2k+2l}} c_i \bar{c}_j \leq$$

$$\begin{aligned}
&\leq \sup_{k,l \in \mathbb{N}_0} \left\{ \frac{a_{k,l}^{\alpha,\beta}}{a_{k,l}^{\gamma,\gamma}} \right\} \sum_{k,l=0}^{\infty} a_{k,l}^{\gamma,\gamma} \sum_{i,j=1}^n \frac{\theta^{k+l} (1-z_i)^k (1-\bar{z}_j)^k (1+z_i)^l (1+\bar{z}_j)^l}{(\theta+1)^{2k+2l}} c_i \bar{c}_j = \\
&= \sup_{k,l \in \mathbb{N}_0} \left\{ \frac{a_{k,l}^{\alpha,\beta}}{a_{k,l}^{\gamma,\gamma}} \right\} \sum_{i,j=1}^n JK_{\gamma,\gamma}^{\theta}(z_i, \bar{z}_j) c_i \bar{c}_j,
\end{aligned}$$

where

$$a_{k,l}^{\alpha,\beta} = \frac{\theta^{\alpha+\beta+1}(\theta-1)}{\tau(\alpha,\beta)(\theta+1)^{\alpha+\beta+2}} \frac{([\alpha+\beta]/2+1)_{k+l}([\alpha+\beta+3]/2)_{k+l}}{(\alpha+1)_k(\beta+1)_l k! l!} > 0.$$

Interchange of the order of summations is justified by absolute convergence of the series (24).

Application of formula (34) yields as $k, l \rightarrow \infty$

$$\frac{a_{k,l}^{\alpha,\beta}}{a_{k,l}^{\gamma,\gamma}} = O\left((k+l)^{\alpha+\beta-2\gamma} k^{\gamma-\alpha} l^{\gamma-\beta}\right) = O\left((1+l/k)^{\alpha-\gamma} (1+k/l)^{\beta-\gamma}\right) = O(1).$$

Therefore $\sup_{k,l \in \mathbb{N}_0} \{a_{k,l}^{\alpha,\beta}/a_{k,l}^{\gamma,\gamma}\}$ is positive and finite. Similarly by choosing $-1 < \eta < \min\{\alpha, \beta\}$ we can prove that

$$mJK_{\eta,\eta}^{\theta} \ll JK_{\alpha,\beta}^{\theta}.$$

It is left to note that $JK_{\beta,\beta}^{\theta} = GK_{\lambda}^{\theta}$, where $\lambda = \beta + 1/2$, and GK_{λ}^{θ} is defined by (27). Now Theorem 3 gives the desired result. \square

When α and/or β belongs to $(-1, -1/2)$ step 3 of the proof of the Theorem 3 breaks and the problem remains open.

Corollary 4.1 *Condition (1) for the Fourier-Jacobi coefficients (4) of a function f is satisfied for all $\theta > 1$ iff f is the restriction of an entire function to the interval $(-1, 1)$.*

The last theorem and Szegő's theory [Szego91] suggest that the following much more general conjecture might be true.

Conjecture. *Inequality (1) holds true for the Fourier coefficients in polynomials orthonormal on $(-1, 1)$ with respect to a weight w that satisfies Szegő's condition $\int_{-1}^1 \ln w(x) dx / \sqrt{1-x^2} > -\infty$ if and only if f belongs to $AL_2(\partial E_{\theta})$.*

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